Analytic Approaches to Single Scattering in Participating Medias

Jakob Gath* Date: 07-2008

Abstract

An analytical approach toward visualization of participating media is presented. The volume rendering equation is considered under the assumption that the in-scattering integral can be sampled as a Dirac delta function in the direction of the illuminating light source. This assumption leads to a completely general result for isotropic homogeneous single scattering medias for which the airlight integral is shown to be a special case of. In addition, the limit for which a light source positioned far away is considered, which is particularly interesting in the case of ocean rendering.

The properties of the result are investigated and an approximate model for point light sources is proposed. The presented model is shown to approximate well for thin medias with a close light source and is suitable for real-time rendering. It is capable of simulating homogeneous translucent medias with a low albedo like marble, wax, skin, and smoke.

Keywords: single scattering, participating media, realistic image synthesis, real-time rendering, translucent rendering, ocean rendering

1 Introduction

Visualizing participating medias are one of the most computationally expensive tasks in realistic image synthesis. This is due to the contribution from multiple scattering inside the media, which in the case of ray marching means recursive evaluation of an already recursive formula. The expensive cost of evaluating multiple scattering is why most simulations of participating medias neglect this contribution, and we shall make no difference in this analytical approach. Neglecting multiple scattering will make medias with high albedo look darker than they are.

Among recent research in real-time computer graphics some innovative examples can be found. For example the CryENGINE 2 shows some innovative images. Although these approaches might not be perfectly correct in a physical sense they produces impressive renderings of medias such as ice, jade, vegetation and skin. Some insight of Crytek's approach can be found in [Wenzel 2007].

Currently, two good real-time rendering methods for simulating participating medias exist. The first one is given in the first book of the GPU Gems series (see [Green 2004]). In this method only direct lighting is taking into account. The second is given in [Sun et al. 2005], where the airlight integral is considered. This is a physical single scattering approach, but is limited to situations where the observer and light is in the same media.

In the first part of the article, the derivation of an general analytic expression for an single scattering media will be considered. The case of a distant light source, which is particularly interesting in the case of ocean rendering, is shown to be a limit and in addition, the airlight integral is shown to be a special case. In the second part, the case of visualizing homogeneous translucent objects illuminated by a near point light source is considered. Finally, an analysis is performed on which an approximative model is based.

*e-mail: hel@blacksmith-studios.dk

2 Analytic Single Scattering

Consider a media illuminated by a point light source positioned at some point $\vec{l}$ being observed from a point $\vec{e}$ along a certain direction. In rendering it is interesting to know exactly how much light the observer will receive from the scattering within the media as a function of this particular direction. A simple situation is depicted in figure 1 where the medium is shown as an infinite plane of finite width. Note that all vectors denoted by $\vec{\omega}$ is of unit length. Since

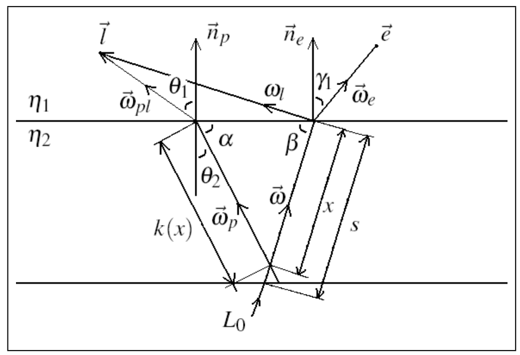

Figure 1: *An infinite plane of participating material illuminated by a point light situated at the point $\vec{l}$ viewed from the virtual eye $\vec{e}$.*

the radiance is constant along a ray, the radiance at the eye along the direction defined by $\vec{\omega}_e$ equals the radiance at the intersection with the surface of the media. It is therefore possible to express the amount of radiance toward the virtual eye by simply using the volume equation (see [Jensen 2001])

$$L(\vec{\omega}) = \int_0^s \exp(-\tau(0,x))\sigma_s(x) \times \int_{\Omega_{4\pi}} p(x,\vec{\omega},\vec{\omega}_p)L_i(x,\vec{\omega}_p)d\vec{\omega}_p dx, \quad (1)$$

where integration is performed over the piece of ray inside the medium. Anything coming from the other side of the medium L_0 and any emission within the medium has not been included. Note that a contribution coming from the other side of the media is trivial to add (see [Green 2004]). The integral over all directions weights the incoming radiances L_i with the phase function p. The optical depth can be defined as:

$$\tau(x_1,x_2) = \int_{x_1}^{x_2} \sigma_t(t)dt$$

where σ_t, is the extinction coefficient defined as the sum of the scattering and absorption coefficient $\sigma_t = \sigma_s + \sigma_a$.

To proceed analytically it is assumed that the medium is throughout homogeneous, in other words $\sigma_t = \text{const.}$.

$$\tau(0,x) = \int_0^x \sigma_t(t)dt = x\sigma_t$$

Assuming that the ray with direction $\vec{\omega}$ at point x through the media only receives light from the refracted path defined by $\vec{\omega}_p$ to the light source the integral over $\Omega_{4\pi}$ collapses to the single direction $\vec{\omega}_p$. It is then possible to write equation 1 as:

$$L(\omega) = \int_0^s \exp(-x\sigma_t)\sigma_s(x)p(x,\vec{\omega},\vec{\omega}_p)L_i(x,\vec{\omega}_p)dx$$

This assumption neglect all multiple scattering and is known as single scattering. Single scattering can be used to approximate medias with low albedo (low scattering, high absorption). The albedo can be written:

$$\Lambda = \frac{\sigma_s}{\sigma_t}$$

Under the assumption of single scattering, the incoming radiance L_i coming from the direction $\vec{\omega}_p$ through the homogeneous media can be evaluated as:

$$\begin{aligned} L_i(x,\vec{\omega}_p) &= \exp(-\tau(0,k(x))L_{\text{refract}}(x) \\ &= \exp(-k(x)\sigma_t)L_{\text{refract}}(x) \end{aligned} \quad (2)$$

where k is the distance from $-x\vec{\omega}$ to the surface of the media in the direction $\vec{\omega}_p$. The $L_{\text{refract}}(x)$ is the amount of refracted radiance coming directly from the light source. It is a function of x in general, since the point where the light strikes the surface moves for different values of x.

Inserting this into the equation for the radiance toward the eye yields:

$$L(\vec{\omega}) = \sigma_s p(\vec{\omega}) \int_0^s \exp(-\sigma_t(x+k(x)))L_{\text{refract}}(x)dx \quad (3)$$

where it is assumed that the phase function does not dependent on the position in the media (homogeneous media). Note that even though the media is assumed to be homogeneous, it is still possible to make the phase function depend on the direction of integration (since it is constant). Furthermore, it should be noted that since the media is assumed to be homogeneous the scattering and absorption coefficients are constant. In the case of isotropic medias the phase function takes the form $p = \frac{1}{4\pi}$.

3 The $k(x)$ Relation

At each position x in the media the distance to the surface toward the point light k will change accordingly. It is easiest to treat the general case, where the surface of the media is refractive. The medium is, therefore, assigned the index of refraction η_2, while the environment is assigned η_1. At the boundary the following relation must hold:

$$\eta_1 \sin\theta_1 = \eta_2 \sin\theta_2$$

Defining $\alpha = \frac{\pi}{2} - \theta_2$ this can be expressed as:

$$\begin{aligned} \eta_1 \sin\theta_1 &= \eta_2 \cos\alpha \\ \eta_1^2(1-\cos^2\theta_1) &= \eta_2^2(1-\sin^2\alpha) \end{aligned}$$

which yields the relation:

$$\sin^2\alpha = 1 - \left(\frac{\eta_1}{\eta_2}\right)^2 (1-\cos^2\theta_1)$$

the cosine term is desirable since $\cos\theta_1 = \vec{\omega}_{pl} \cdot \vec{n}_p$. In the trivial case where $\eta_1 = \eta_2$ the expression simplifies to:

$$\sin\alpha = \cos\theta_1 = \vec{\omega}_{pl} \cdot \vec{n}_p,$$

For the surface intersection along the eye direction a similar relation can be found:

$$\sin^2\beta = 1 - \left(\frac{\eta_1}{\eta_2}\right)^2 (1-\cos^2\gamma_1)$$

where $\cos\gamma_1 = \vec{\omega}_e \cdot \vec{n}_e$.

It is important to note that if the media is refractive the direction $\vec{\omega}_{pl}$ is the refracted direction from the light source. In this case we have to use the law of sines:

$$k = \frac{\sin\beta}{\sin\alpha} x = cx$$

where c is some constant, depending on the index of refraction one may choose. In the case of $\eta_1 = \eta_2$ this can be written:

$$k(x) = \frac{\vec{\omega}_e \cdot \vec{n}_e}{\vec{\omega}_{pl} \cdot \vec{n}_p} x \quad (4)$$

It is important to remember that $\vec{\omega}_{pl}$ and $\vec{n}_p$ are a function of x.

3.1 Distant Light Source

If the light source is positioned far away from the ray which is scattering the light backwards to the eye, it is possible to assume that the direction toward the light source can be seen as independent of x, thereby having a constant direction $\vec{\omega}_{pl}(x) = \vec{\omega}_{pl}$. This assumption is valid as long as the length of integration s is short compared to the distance to the light. If the normal at the intersection of the light ray in addition is assumed to be constant the relation c will be a constant. When the light source is positioned far away, it is also possible to assume that the radiance coming from the light source is constant along the eye ray $L_{\text{refract}} = \text{const.}$. Using these assumptions and inserting the relation 4 into equation 3 one obtains:

$$\begin{aligned} L(\vec{\omega}) &= \sigma_s p(\vec{\omega}) L_{\text{refract}} \int_0^s \exp(-\sigma_t x(1+c))dx \\ &= \Lambda \frac{p(\vec{\omega})}{1+c}(1-\exp(-\sigma_t(1+c)s)) L_{\text{refract}} \end{aligned} \quad (5)$$

Where s is the length of the integration through the media.

This expression could be very close to what Crytek are using to simulate their ocean rendering and underwater volumetric effects. Note that in the case of the eye being positioned under the surface the integration changes.

Let us investigate the properties of the result. Letting s tend to zero the radiance also tend to zero as expected, since this means that there is not any media to scatter the light from the light source. On the other hand letting s tend to infinity yields:

$$L(\vec{\omega}) = \Lambda \frac{p(\vec{\omega},\vec{\omega}_{pl})}{(1+c)} L_{\text{refract}}$$

This limit is ideally wrong under the assumption that the light source is located far away, since the length of integration is now infinite. The direction toward the light source can therefore not be assumed to be constant. However, s can still be large under the assumption of $l > s$ which will make the exponential decay present in equation 5 close to zero. The limit is therefore a rough approximation for deep oceans where the depth is large, but the distance to the sun is even larger.

3.2 Near Light Source

In this section the light source is located near the surface making the direction toward the light dependent of the integration variable $\vec{\omega}_p = \vec{\omega}_p(x)$. Ideally the surface normal $\vec{n}_p$ should also be a function of x for allowing arbitrary geometry of the participating media. In the following it is assumed that the surface normal is constant $\vec{n}_p = \vec{n}_e$. For simplicity the media is assumed to be non-refractive $\eta_1 = \eta_2$. In this case, the direction of integration is equal to the direction toward the eye $\vec{\omega} = \vec{\omega}_e$ and the refractive direction toward the light source points exactly toward the light source $\vec{\omega}_p = \vec{\omega}_{pl}$.

Defining $\vec{x} = -x\vec{\omega}$, the direction toward the light can be expressed as:

$$\vec{\omega}_p(x) = \frac{\vec{l} - \vec{x}}{||\vec{l} - \vec{x}||}$$

The denominator can be expressed using the cosine relation:

$$h(x)^2 = ||\vec{l} - \vec{x}||^2 = |x|^2 + |l|^2 + 2|x||l|(\vec{\omega} \cdot \vec{\omega}_l)$$

Note that the last sign is correct in relation to the direction of $\vec{\omega}$. Noting that $\vec{l} - \vec{x}$ can be written $l\vec{\omega}_l + x\vec{\omega}_e$ and provided the assumption of a non-refractive media the new relation of $\omega_p(x)$ can be used in equation 4:

$$k(x) = \frac{x\,(\vec{\omega}_e \cdot \vec{n})\,h(x)}{(l\vec{\omega}_l + x\vec{\omega}_e) \cdot \vec{n}} \tag{6}$$

Note that in the case of refractive medias one could assume that only the direction toward the eye is refracted and thereby use $\sin\beta$ instead of using $\vec{\omega}_e \cdot \vec{n}_e$ in equation 4 and 6.

The relation can be inserted in the single scattering expression given by equation 3:

$$\begin{aligned} L(\vec{\omega}) \quad &= \quad \sigma_s p(\vec{\omega}) \int_0^s L(x) \exp\left(-\sigma_t x\right) \\ &\quad \times \exp\left(-\sigma_t \frac{x\,(\vec{\omega}_e \cdot \vec{n})\,h(x)}{(l\vec{\omega}_l + x\vec{\omega}_e) \cdot \vec{n}}\right) dx \end{aligned} \tag{7}$$

This integral is the analytic expression for computing the radiance toward the eye from a piece of scattering homogeneous media which only scatter the light it receives a single time.

Rewriting the integral by expressing the exponential function with its infinite sum, it is possible to write the expression as a series of integrals over polynomials. The sum and integral can be interchanged because the series is uniformly convergent. The minus sign results in an alternating series. Unfortunately, it is only possible to evaluate the first few terms of the series analytical, which could be used as an n'th order approximation, but even these are far to computationally expensive to use for fast rendering.

Note that using the same approach as [Sun et al. 2005] to rewrite the expression 7 lead to the need of a three-dimension lookup table. One can follow that idea, but it is however more interesting to find an approximative method, which is easy and fast to evaluate. This is the goal of the the next sections by investigating the behavior of equation 7.

3.3 Point Light Sources

In real-time rendering one is often using the approximation of point lights. Point lights can be used under the assumption that the medium which the light is traveling in is homogeneous in all directions and there are no absorption. In the previous sections the derivation has been based on homogeneous scattering medias and absorption within the medium is present in equation 2. The radiance of a point source decreases in proportion to the inverse square of the distance

$$L_i(x) = \frac{\Phi}{4\pi h(x)^2} = \frac{I_0}{h(x)^2},$$

where Φ is the radiant power of the light source and I_0 the radiant intensity defined by $I_0 = \frac{\Phi}{4\pi}$. These expressions can be inserted directly into equation 7. One should keep in mind that point lights are not physical correct and their density fail when close to the illuminated medium.

3.4 Relation to the Airlight Integral

The airlight integral [Sun et al. 2005] describes the incoming radiance when the eye and light source both are present in the media. In order to reach the airlight integral one should constraint the position of the light in the plane of the media such that the light always travels within the participating media. In the previous section the light was first traveling in air and then in the media.

We constrain the position of the light to be in the plane, say $l_z = 0$. Now since $\vec{n}$ only have a z component the dot product $\vec{l} \cdot \vec{n}_l$ will be zero. This reduces the integral given by 7 to the airlight integral:

$$L(\vec{\omega}) = I_0 \sigma_s p(\vec{\omega}) \int_0^s \frac{\exp\left(-\sigma_t(x + h(x))\right)}{h(x)^2} dx \tag{8}$$

For which a simple solution exist (see [Sun et al. 2005]), which are easily implemented for real-time rendering. Although one should note that [Sun et al. 2005] are using $\sigma_s = \sigma_t$.

The solution is worth stating in terms of the current parameters. Defining the angle:

$$\begin{aligned} \gamma \quad &= \quad \pi - \arccos(-\vec{\omega} \cdot \vec{\omega}_l) \\ &= \quad \arccos(\vec{\omega} \cdot \vec{\omega}_l) \end{aligned}$$

The solution can be written as:

$$\begin{aligned} A_0 \quad &= \quad 2I_0 \sigma_s p(\vec{\omega}) \frac{\exp(-\sigma_t l \cos(\gamma))}{l \sin\gamma} \\ A_1 \quad &= \quad \sigma_t l \sin(\gamma) \\ F(u,v) \quad &= \quad \int_0^v \exp(-u \tan\xi) d\xi \\ v_1 \quad &= \quad \frac{\pi}{4} + \frac{1}{2} \arctan\left(\frac{s - l\cos\gamma}{l \sin\gamma}\right) \\ v_2 \quad &= \quad \frac{\gamma}{2} \\ L \quad &= \quad A_0(F(A_1, v_1) - F(A_1, v_2)) \end{aligned}$$

The analytical solution to the function $F(u,v)$ involves elliptic functions. A lookup table has therefore been used by [Sun et al. 2005] in order to evaluate it.

4 Results

4.1 Comparison

In this section the single scattering media integral 7 and airlight integral 8 are investigated under the assumption of a point light source. The presences of the $h(x)^2$ falloff in both integrals are due to the point light. In order to compare the expressions the fall off is removed from the arguments of the integrals. This approach also makes it possible to compare with the limit of the light source being positioned far away given by equation 5.

The general behavior of the integral 7 will be that for a finite fixed s it will converge to the limit given by equation 5 for increasing l, while the airlight integral will go to zero.

On figure 2, the integrals are shown as a function of the distance to the light source l for the case of $c = 1$. Both the airlight and the single scattering media integrals are greater than the limit for small l. However for larger values of l the airlight integral approaches zero while the single scattering media integral converges to the limit of the light source being far away. One thing to note is the coincidence in the case of $\vec{\omega}_e \cdot \vec{n}_e = \vec{\omega}_{pl} \cdot \vec{n}_p = 1$ where the integral 7 reduces exactly to the limit value because $l + x = \sqrt{x^2 + l^2 + 2lx}$.

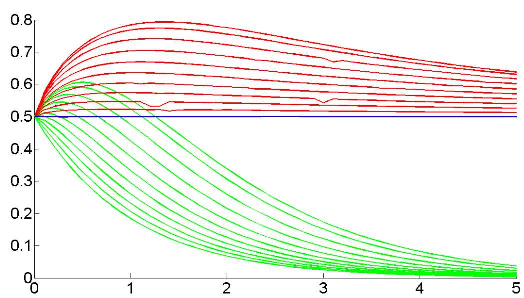

Figure 2: *The two integrals as a function of the distance to the light (**red**: single scattering media, **green**: airlight, **blue**: $l = \infty$ limit). Parameters: $s = 10$, $c = 1$, $\sigma_t = 1$. Note that for $\vec{\omega}_e \cdot \vec{n}_e = \vec{\omega}_{pl} \cdot \vec{n}_p = 1$ the single scattering media coincidence with the $l = \infty$ limit.*

On figure 3, the single scattering media integral is shown for different relations between the light and eye position together with the limit value given by equation 5. Two things are important to note in the case of $c = \infty$ where $\vec{\omega}_{pl} \cdot \vec{n}_p = 0$. Firstly, the light vector is parallel with the surface for which it is expected that the limit given by equation 5 is zero and secondly, the single scattering media integral reduces to the airlight integral (shown by a green plot line).

For thick medias where s is likely to be large, it can be shown that the integral over the exponential part of equation 7 decreases more slowly. Furthermore, it is evident that medias with small extinction coefficients will result in a slower decreasing exponential part. Figure 6 and 7 in appendix A illustrates to some degree these behaviors.

4.2 Approximation and Error Analysis

Including the falloff of $h(x)^2$ in the integrals will make the shape of the functions seem very similar due to the relatively quickly converging exponential part observed in figure 2 and 3. From section 4.1 where the integral over the exponential function was investigated, it was mentioned that when the media is rather thin (s small)

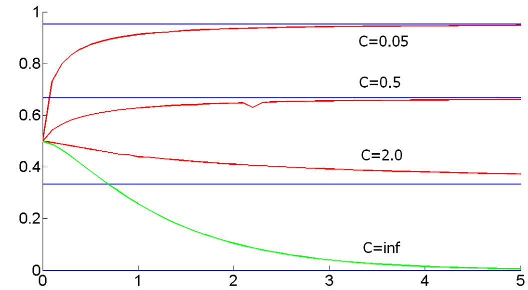

Figure 3: *The single scattering media integral as a function of the distance to the light for different values of c. Parameters: $s = 10$, $\sigma_t = 1$.*

the integral over the exponential function tends to be approximative constant as a function of l. In the case of such medias, it is therefore a good approximation to use an average of the integral over the exponential part. It is reasonable to use the asymptotic limit given by equation 5 to estimate the average. The average value is obtained by dividing by the integration lenght s. The radiance is then approximated by:

$$L_{\text{approx}}(\vec{\omega}) = \frac{I_0}{s} \frac{\sigma_s\, p(\vec{\omega})}{\sigma_t(1+c)} \left(1 - \exp\left(-\sigma_t(1+c)s\right)\right) \int_0^s h(x)^{-2} dx$$

The integral over $h(x)^{-2}$ can be computed analytically. Let $g = \vec{\omega} \cdot \vec{\omega}_l$:

$$\frac{1}{l\sqrt{1-g^2}} \left(\arctan\left(\frac{s + lg}{l\sqrt{1-g^2}} \right) - \arctan\left(\frac{g}{\sqrt{1-g^2}} \right) \right)$$

The integral is seen to be defined for the domains of interest of the parameters: $l > 0$, $s > 0$, $g = [0;1[$. Note that $\arctan : \mathbb{R} \to]-\frac{\pi}{2};\frac{\pi}{2}[$. However for the special limit of $g = 1$ the integral converges to the limit:

$$\int_0^s \frac{1}{x^2 + l^2 + 2xl} dx \quad = \quad \frac{s}{l(s+l)}$$

The integral over $h(x)^2$ has a singularity for $l = 0$ in which it increases toward infinity as l tend to zero. A deviation in the average value for the exponential part will therefore result in a large deviation of the total result. The approximation will therefore inevitable break down when the light source is located close to the medium (really small values of l). However, in this case the approximation of a point light is also wrong.

Figure 4 shows some results for the exact single scattering integral and the discussed approximation. The results are for a relative thin media where s is small. The corresponding error estimates are shown in figure 5. Considering figure 8 in appendix A, it is evident that when the media is thick (s tend to be large as well) the integral over the exponential function is of greater importance for small l. It is in this situation that the approximation breaks down.

5 Conclusion

An analytical expression for the evaluation of radiance coming toward a virtual observer from a given direction from a homogeneous single scattering participating media illuminated by a light source has been derived. The expression have been investigated in the approximation of a point light source and compared to some extent

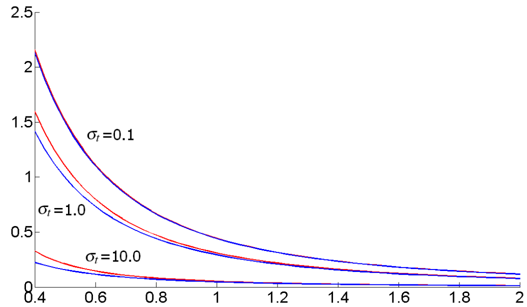

Figure 4: *The single scattering integral and the discussed approximation as a function of l. Parameters:* $\sigma_t = [0.1, 1.0, 10.0], c = 0.7/0.7 = 1, s = 0.5]$.

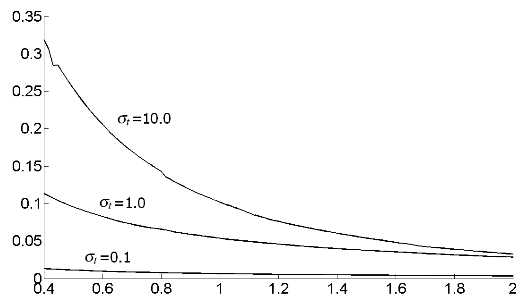

Figure 5: *The error associated with the approximation in relation to the single scattering integral. Parameters:* $\sigma_t = [0.1, 1.0, 10.0], c = 0.7/0.7 = 1, s = 0.5$.

with the airlight integral. The analysis contain some interesting limits as when there is a coincidence with the airlight integral and the case of an light source positioned far away. Based upon the analysis an easy-to-evaluate approximation has been proposed, which has been proven to be valid for thin medias, where the integral over the decaying exponential part as a function of the distance to the light source behaves constantly.

The single scattering approximations considered are for medias with low albedo and are derived mainly with isotropic medias in mind. The analytical expression derived is targeted for fast rendering of participating medias and is most suitable for offline renders while the proposed easy-to-evaluate approximation is for real-time rendering.

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A Illustrations

This appendix contains some additional illustrations used in the article to explain various mathematical properties.

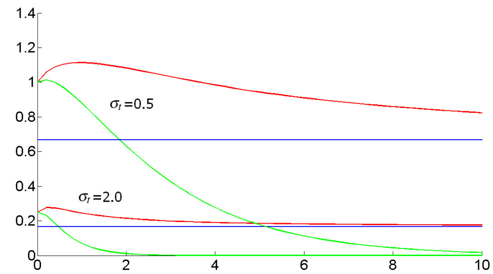

Figure 6: *The two integrals as a function of the distance to the light for a rather thick media* $s = 10$*. Parameters:* $c = 2$.

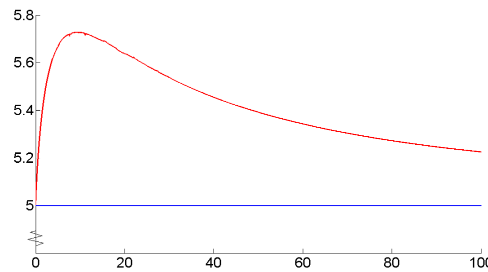

Figure 7: *The single scattering integral and its limit* $l \to \infty$ *as a function of l showing that for thick medias the integral over the exponential function decreases more slowly (note the range of the abscissa). The parameters were* $c = 0.7/0.7 = 1, s = 100, \sigma_t = 0.1$

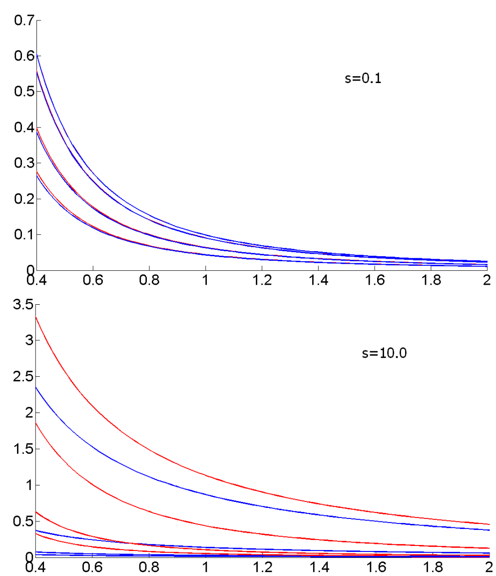

Figure 8: *The single scattering integral and the discussed approximation as a function of l. Parameters:* $\sigma_t = [0.1, 1.0, 10.0], c = 0.7/0.7 = 1, s = [0.1, 10.0]$.